

Essential surfaces in graph pairs

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Abstract

A well known question of Gromov asks whether every one-ended hyperbolic group Γ has a surface subgroup. We give a positive answer when Γ is the fundamental group of a graph of free groups with cyclic edge groups. As a result, Gromov's question is reduced (modulo a technical assumption on 2-torsion) to the case when Γ is rigid. We also find surface subgroups in limit groups. It follows that a limit group with the same profinite completion as a free group must in fact be free, which answers a question of Remeslennikov in this case.

This paper addresses a well known question about hyperbolic groups, usually attributed to Gromov.

Question 0.1. *Does every one-ended hyperbolic group contain a surface subgroup?*

Here, a *surface subgroup* is a subgroup isomorphic to the fundamental group of a closed surface of non-positive Euler characteristic. Various motivations for Gromov's question can be given. It generalizes the famous Surface Subgroup conjecture for hyperbolic 3-manifolds, but it is also a natural challenge when one considers that the Ping-Pong lemma makes free subgroups very easy to construct in hyperbolic groups, whereas a theorem of Gromov–Sela–Delzant [17] asserts that a one-ended group has at most finitely many images in a hyperbolic group. More recently, Markovic proposed finding surface subgroups as a route to proving the Cannon conjecture [31].

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Several important cases of Gromov’s question have recently been resolved. Most famously, Kahn and Markovic proved the Surface Subgroup conjecture [23]. Extending their work has been the topic of a great deal of recent research (see [21, 28], for instance). In another dramatic development, Calegari and Walker answered Gromov’s question affirmatively for random groups [10], following similar results for random ascending HNN extensions of free groups (by the same authors [11]) and random graphs of free groups with edge groups of rank at least two (by Calegari and the author [12]).

In this paper, we resolve Gromov’s question for a contrasting class of hyperbolic groups – graphs of free groups with cyclic edge groups. Our main theorem answers Gromov’s question affirmatively in this case.

Theorem A. *Let Γ be the fundamental group of a graph of free groups with cyclic edge groups. If Γ is one-ended and hyperbolic then Γ contains a quasiconvex surface subgroup.*

In fact, using a result of Wise, we are able to find surface subgroups in graphs of *virtually* free groups with *virtually* cyclic edge groups; see Theorem 6.1 below.

Numerous special cases of this result are already known. Calegari used his work on the rationality of stable commutator length in free groups [9] to show that surface subgroups exist when $H_2(\Gamma; \mathbb{Q}) \neq 0$ [7]. Infinite classes of examples were found by the author in joint works with Gordon [18] and with Kim [27]. Kim and Oum found surface subgroups in doubles of free groups of rank two [26]. The author answered a weaker version of Gromov’s question for this class of groups, by showing that every such Γ is either a surface group or contains a finitely generated, one-ended subgroup of infinite index [42].

Although the class of hyperbolic groups covered by Theorem A is quite specific, the theorem has wider consequences for Gromov’s question. We call a group *rigid* if it does not admit a non-trivial splitting with a virtually cyclic edge group. Using strong accessibility [29] we can, modulo a technical hypothesis on 2-torsion, reduce Gromov’s question to the rigid case, using the following corollary.

Corollary B. *Let Γ be a one-ended hyperbolic group without 2-torsion. Either Γ contains a quasiconvex surface subgroup, or Γ contains a quasiconvex rigid subgroup.*

See Corollary 6.4 for full details. By a theorem of Bowditch [4], a one-ended hyperbolic group Γ is rigid if and only if its Gromov boundary does not

contain local cut points (unless Γ is a finite extension of a triangle group). Corollary B should be useful in any attempt at a general answer to Gromov's question, since local cut points in the boundary present extra technical challenges for the ergodic techniques of [23] and the probabilistic techniques of [10], as witnessed by the difficulties resolved in Kahn and Markovic's proof of the Ehrenpreis conjecture [24].

A *limit group* is a finitely generated, fully residually free group – that is, a finitely generated group in which every finite subset can be mapped injectively into a free group by a group homomorphism. Limit groups play a central role in the study of algebraic geometry and logic over free groups; see [36] *et seq.*, in which they were defined, and also the parallel project [25] *et seq.* Theorem A addresses the key case for the problem of finding surface subgroups of limit groups, and so we can also answer Gromov's question in that context.

Corollary C. *Let Γ be a limit group. If Γ is one-ended then Γ contains a surface subgroup.*

See Corollary 6.3 for details. Note that limit groups are not all hyperbolic, but they are all toral relatively hyperbolic [1, 16]. In particular, non-hyperbolic limit groups contain a \mathbb{Z}^2 subgroup, so the hyperbolic case is the one of interest.

These results have interesting applications to a different structural problem in group theory. Recall that the *profinite completion*, $\widehat{\Gamma}$, of a group Γ is the closure of the image of Γ in the direct product of its finite quotients (endowed with the product topology). If two groups Γ_1 and Γ_2 have isomorphic profinite completions, then it is natural to ask whether Γ_1 and Γ_2 must be isomorphic.

In general, the answer is ‘no’. There are even examples of non-isomorphic pairs of virtually cyclic groups with isomorphic profinite completions [2]. Nevertheless, many important questions of this type remain open, of which the following question of Remeslennikov is one of the most notable [32, Question 15].

Question 0.2 (Remeslennikov). *Suppose that F is a finitely generated, non-abelian free group and that Γ is finitely generated and residually finite. If $\widehat{\Gamma} \cong \widehat{F}$, does it follow that $\Gamma \cong F$?*

It is particularly natural to consider Question 0.2 when Γ is a limit group. Indeed, limit groups are closely related to free groups (for instance, Remeslen-

nikov showed that they are precisely the *existentially free groups* [34]), and are frequently hard to distinguish from them. Bridson, Conder and Reid [6] pointed out that Corollary C, combined with the results of [41], would resolve Remeslennikov's question in this case.

Corollary D. *If L is a limit group and not free then the profinite completion \widehat{L} is not isomorphic to the profinite completion of any free group.*

The same ideas give a new proof of a theorem of Puder and Parzanchevski [33, Corollary 1.5]. Recall that a word w in a free group F is called *primitive* if F splits as a free product $\langle w \rangle * F_1$. Similarly, an element \hat{w} of the profinite free group \widehat{F} is called *primitive* if \widehat{F} decomposes as a coproduct $\overline{\langle \hat{w} \rangle} \coprod \widehat{F}_1$ in the category of profinite groups. Puder and Parzanchevski showed that an element w of F that is primitive in the profinite completion \widehat{F} is already primitive in F [33, Corollary 1.5]. This can be thought of as answering a relative version of Question 0.2.

In fact, we can generalize their result, from words to multiwords (i.e. finite indexed sets of words). Let us call a multiword $\underline{w} = \{w_1, \dots, w_n\}$ in F *primitive* if F splits as a free product $\langle w_1 \rangle * \dots * \langle w_n \rangle * F_1$ for some F_1 , and make the corresponding definition of a primitive multiword in \widehat{F} .

Corollary E. *Let F be a finitely generated free group. If a multiword \underline{w} is primitive in the profinite completion \widehat{F} then it is primitive in F .*

It is interesting to contrast the techniques of this paper with those of [33]. Puder and Parzanchevski deduce their result from their beautiful characterization of primitive words in free groups as precisely the *measure-preserving words* [33, Theorem 1.1]. The proof given here is cohomological, and goes via the fact that the virtual second cohomology of the profinite completion of a non-free limit group is non-zero. Corollaries D and E follow quickly from Theorem 7.1. We refer the reader to that theorem and the subsequent remarks for details.

Let us now turn to discuss the proof of Theorem A. Our main technical result addresses a *relative* version of Gromov's question, finding surfaces in free groups relative to families of cyclic subgroups. To state it concisely we need to introduce some definitions.

Consider a graph of spaces X in the sense of Scott and Wall [35], and let v be a vertex with incident edges e_1, \dots, e_n . The vertex space X_v , together with the maps of incident edge spaces $w_i : X_{e_i} \rightarrow X_v$, defines a *space pair*

(X_v, \underline{w}) . In the case of interest, the vertex space X_v will always be a graph (usually denoted by Γ), and the edge spaces X_{e_i} will be circles S_i^1 ; such a (Γ, \underline{w}) is called a *graph pair*.

Global properties of the graph of spaces X can be characterized locally, using properties of the pairs associated to vertex spaces. For instance, X is called *irreducible* if $\pi_1 X$ does not split over a finite subgroup; we may correspondingly define an *irreducible pair* (Γ, \underline{w}) (see Definition 2.9), and a lemma of Shenitzer asserts that if the pairs associated to the vertices are irreducible, then so is X [42, Theorem 18]. Corresponding to the notion of a π_1 -injective map of graphs of spaces $Y \rightarrow X$, we have an *essential* map of pairs $(\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$, and indeed if a morphism of graphs of spaces is essential on each space pair associated to a vertex then the morphism is itself π_1 -injective (see Proposition 1.7).

We are now ready to state the main technical result.

Theorem F. *If (Γ, \underline{w}) is an irreducible graph pair then there is a compact surface with boundary Σ and an essential map of pairs $(\Sigma, \partial\Sigma) \rightarrow (\Gamma, \underline{w})$.*

In fact, we obtain a bit more control than this – the surface is also *admissible*, meaning that every point of the domain of \underline{w} has the same number of preimages in $\partial\Sigma$; see Theorem 5.11. It has been well known for a while that a result like Theorem F would imply the existence of surface subgroups in graphs of free groups with cyclic edge groups – see, for instance, [7] or [27].

Let us now briefly sketch the proof of Theorem F. It can be thought of as a combination of the techniques of [9] and [42].

First, we study irreducible pairs (Λ, \underline{u}) that map into the irreducible pair (Γ, \underline{w}) . The irreducibility of the pairs (Γ, \underline{w}) and (Λ, \underline{u}) is characterized using *Whitehead graphs*. We would like to study essential maps $(\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$, but it turns out to be difficult to characterize essential maps of irreducible pairs. Instead, we work with *∂ -essential* maps, meaning that the lift of the map $(\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$ to the universal cover of Γ doesn't fold edge spaces together. These can be efficiently characterized using *∂ -immersions* (see Definition 1.19).

The idea behind the proof of Theorem F is that, among all irreducible pairs mapping to (Γ, \underline{w}) , the pairs of surface type should be the ones of most negative Euler characteristic. To make this precise, we define a positive polyhedral cone $C_{\mathcal{P}}$ in a finite-dimensional vector space, such that the integer points in $C_{\mathcal{P}}$ correspond to admissible ∂ -immersions of irreducible pairs

(Λ, \underline{u}) . We also define the *projective \mathcal{P} -rank function* $\rho_{\mathcal{P}}$ on the projectivization $\mathbb{P}(C_{\mathcal{P}})$ as a quotient of two linear functionals: the (negation of the) Euler characteristic of Λ , and the degree with which \underline{u} covers \underline{w} . In particular, $\rho_{\mathcal{P}}$ achieves its maximum value at some vertex of the polyhedron $\mathbb{P}(C_{\mathcal{P}})$, which is necessarily a rational line in $C_{\mathcal{P}}$. Since this rational line contains an integer point, an admissible ∂ -immersion of an irreducible pair exists that maximizes $\rho_{\mathcal{P}}$. We call such a pair (Λ, \underline{u}) *maximal*.

This approach is similar to the argument of [9], in which a polyhedral cone is defined whose integer points correspond to certain maps of surfaces $(\Sigma, \partial\Sigma) \rightarrow (\Gamma, \underline{w})$. The hypothesis in [7] that rational second homology is non-zero is needed to ensure that this cone is non-zero. In contrast, the cone $C_{\mathcal{P}}$ is guaranteed to be non-zero since, whenever (Γ, \underline{w}) is irreducible, the identity map $(\Gamma, \underline{w}) \rightarrow (\Gamma, \underline{w})$ leads to an admissible ∂ -immersion of an irreducible pair.

The final step of the proof applies the ideas of [42] to the relative JSJ decomposition of a maximal pair (Λ, \underline{u}) . The conclusion is that any maximal pair has no rigid vertices in its JSJ decomposition. It follows that the JSJ decomposition is built from surface pieces, and one quickly concludes that a maximal pair of surface type exists. Thus, we deduce the existence of an admissible, ∂ -essential surface $(\Sigma, \partial\Sigma) \rightarrow (\Gamma, \underline{w})$. To upgrade this to an essential surface, we appeal to a theorem of Calegari (see Theorem 4.5), which shows that we can promote a ∂ -essential map of a surface to an essential one.

The paper is structured as follows. In Section 1, we define pairs of groups, spaces and graphs, the natural notions of maps between them, and various properties of those maps. In Section 2 we adapt the classical theory of Whitehead graphs to the setting of a graph pair (Γ, \underline{w}) . The main result here is a converse to Whitehead's lemma (Lemma 2.11), which asserts that an irreducible pair can always be unfolded to a *locally* irreducible pair, in which the irreducibility is recognized by the Whitehead graphs at the vertices. In Section 3, we characterize admissible ∂ -immersions from locally irreducible graph pairs into (Γ, \underline{w}) as precisely those maps that can be built from a certain finite set \mathcal{P} of *pieces*. We define the cone $C_{\mathcal{P}}$ and note that there is a surjective map from admissible ∂ -immersions of locally irreducible pairs to the integer points of $C_{\mathcal{P}}$. In Section 4, we define the *projective \mathcal{P} -rank function* $\rho_{\mathcal{P}}$, and prove that it attains its extremal values at rational points of $\mathbb{P}(C_{\mathcal{P}})$. We deduce the existence of a maximal, admissible ∂ -immersion from a locally irreducible pair. In Section 5, we apply the results of [42] to study admissible ∂ -immersions with maximal projective \mathcal{P} -rank. The main

result is that there is such a maximal pair of surface type (Theorem 5.10), which completes the proof of Theorem F. In Section 6 we deduce Theorem A and Corollaries B and C. Finally, in Section 7, we deduce Corollaries D and E.

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1 Pairs

We will make heavy use of graphs of groups and Bass–Serre theory, as detailed in Serre’s standard work on the subject [37], to which the reader is referred for details. To fix notation, we recall the definition of a graph.

Definition 1.1. A *graph* Γ consists of a vertex set V , an edge set E , a fixed-point free involution $E \rightarrow E$ denoted by $e \mapsto \bar{e}$, and an *origin map* $\iota : E \rightarrow V$. The *terminus map* $\tau : E \rightarrow V$ is defined by $\tau(e) = \iota(\bar{e})$.

The edges of Γ are thus equipped with orientations, and the unoriented edges are the pairs $\{e, \bar{e}\}$.

As well as using graphs of groups, we will also frequently adopt the topological point of view, in which a graph of groups is viewed as the fundamental group of a graph of spaces [35]. Graphs of spaces are not required to be connected, which will present some technical advantages, although the attaching maps are required to be injective on fundamental groups. Analogously, we may also work with disconnected graphs of groups, as long as we are careful to choose a base point before talking about the fundamental group.

1.1 Group pairs

It is particularly important for us to work with *relative* versions of graphs of groups and spaces, which characterize the relationship between a vertex group (or space) and its incident edge groups (or spaces). To this end, we define various notions of pairs. We start with pairs of groups.

Definition 1.2. A *group pair* is a pair (G, \mathcal{A}) , where G is a group and \mathcal{A} is a G -set. It is often convenient to choose a finite set of orbit representatives $\{a_i\}$, to let $H_i = \text{Stab}_G(a_i)$, and to specify the pair via the data $(G, \{H_i\})$. We will use both the notations (G, \mathcal{A}) and $(G, \{H_i\})$ to specify group pairs, without fear of confusion.

The key example of a group pair arises when considering a vertex v of a graph of groups \mathcal{G} . Having fixed a lift \tilde{v} of v to the Bass–Serre tree, one takes G to be the vertex stabilizer $\mathcal{G}_{\tilde{v}}$ and \mathcal{A} to be the set of edges incident at \tilde{v} .

Definition 1.3. A *morphism of graphs of groups* is a morphism of the underlying graphs, accompanied by associated maps of vertex groups and edge groups that intertwine with the attaching maps. This is most easily thought of by passing to the Bass–Serre tree. A morphism of graphs of groups induces a homomorphism of fundamental groups, and lifts to an equivariant map on Bass–Serre trees.

This motivates the following definition for pairs.

Definition 1.4. A *morphism of group pairs* $(f, \phi) : (G, \mathcal{A}) \rightarrow (G', \mathcal{A}')$ consists of a set map $\phi : \mathcal{A} \rightarrow \mathcal{A}'$ and a homomorphism $f : G \rightarrow G'$ that intertwines ϕ . That is, we require that

$$\phi(g.a) = f(g).\phi(a)$$

for all $g \in G$ and $a \in \mathcal{A}$.

In particular, a morphism of graphs of groups defines morphisms of the group pairs at each vertex, and conversely morphisms of pairs that satisfy an obvious compatibility condition can be pieced together to give a morphism of a graph of groups.

It is convenient if we can detect *global* properties of morphisms of graphs of groups by looking at *local* properties of the induced maps on group pairs. We are particularly concerned with π_1 -injectivity, and so we need to develop

corresponding notions for group pairs. Requiring that the map of groups $f : G \rightarrow G'$ be injective is clearly significant. The following condition is also important.

Definition 1.5. A morphism of group pairs $(f, \phi) : (G, \mathcal{A}) \rightarrow (G', \mathcal{A}')$ is *∂ -essential* if the map

$$\ker f \backslash \mathcal{A} \rightarrow \mathcal{A}'$$

induced by ϕ is injective.

When applied to pairs associated to graphs of groups, this condition guarantees that the induced map on Bass–Serre trees does not factor through a fold. Putting this together with injectivity, we have the notion of an essential morphism.

Definition 1.6. A morphism of group pairs $(f, \phi) : (G, \mathcal{A}) \rightarrow (G', \mathcal{A}')$ is *essential* if the homomorphism $f : G \rightarrow G'$ is injective and the morphism is also ∂ -essential.

From this one easily deduces a local criterion for morphisms of graphs of groups to be π_1 -injective.

Proposition 1.7. *Suppose that f is a morphism of graphs of groups. If f induces essential morphisms on the group pairs corresponding to vertices, then f induces an injective map on fundamental groups.*

Proof. Suppose that a group element g is in the kernel of f . Since the map on Bass–Serre trees does not factor through a fold, if g acts hyperbolically on the Bass–Serre tree then so does its image, contradicting the fact that g is in the kernel. Therefore g is elliptic, but since f is injective on vertex stabilizers, it follows that $g = 1$. \square

1.2 Space pairs

We next make analogous definitions for spaces.

Definition 1.8. A *space pair* consists of cell complexes X, Y together with a continuous map $\underline{w} : Y \rightarrow X$. We will frequently take $\pi_0 Y$ to be an index set I , and let

$$w_i : Y_i \rightarrow X$$

denote the restriction of \underline{w} to Y_i , the path component of Y corresponding to $i \in I$. We will often use the notation (X, \underline{w}) to denote such a space pair.

In most of what follows, we will take X to be a graph and Y to be a disjoint union of circles. However, it is useful to allow the extra flexibility of the general definition.

If X is path connected then a space pair (X, \underline{w}) naturally defines a group pair (G, \mathcal{A}) . Let $p : \tilde{X} \rightarrow X$ be the universal cover, and consider the fibre product

$$\tilde{Y} := \tilde{X} \times_X Y = \{(\tilde{x}, y) \mid p(\tilde{x}) = \underline{w}(y)\} .$$

Taking $G = \pi_1 X$ and $\mathcal{A} = \pi_0 \tilde{Y}$, we see that G acts naturally on \mathcal{A} , and so (G, \mathcal{A}) is a group pair.

This definition is more transparent if one thinks of X as a vertex space of a graph of spaces Z . The universal cover \tilde{Z} of Z inherits a decomposition as a graph of spaces; \tilde{X} appears as a vertex space of \tilde{Z} , and the fibre product \tilde{Y} is the disjoint union of the edge spaces of \tilde{Z} incident at \tilde{X} .

We next define morphisms of space pairs, analogously to morphisms of group pairs.

Definition 1.9. Let $\underline{w} : Y \rightarrow X$ and $\underline{w}' : Y' \rightarrow X'$ define space pairs (X, \underline{w}) and (X', \underline{w}') . A *morphism of space pairs* $(X, \underline{w}) \rightarrow (X', \underline{w}')$ consists of continuous maps $\phi : Y \rightarrow Y'$ and $f : X \rightarrow X'$ so that $f \circ \underline{w} = \underline{w}' \circ \phi$.

As in the case of groups, compatible collections of maps of pairs can be glued together to construct a map of graphs of spaces. Again, we will need a definition of a ∂ -essential morphism.

Definition 1.10. Consider a morphism of space pairs $(X, \underline{w}) \rightarrow (X', \underline{w}')$. Let \tilde{X}' be the universal cover of X' and let \hat{X} be the corresponding covering space of X , obtained by pulling back the covering map $\tilde{X}' \rightarrow X'$ along f . Consider the fibre products

$$\hat{Y} = \hat{X} \times_X Y , \quad \tilde{Y}' = \tilde{X}' \times_{X'} Y' ,$$

and note that the map $\phi : Y \rightarrow Y'$ lifts to a map $\tilde{\phi} : \hat{Y} \rightarrow \tilde{Y}'$. The morphism $(X, \underline{w}) \rightarrow (X', \underline{w}')$ is called *∂ -essential* if $\tilde{\phi}$ induces an injective map $\pi_0 \hat{Y} \rightarrow \pi_0 \tilde{Y}'$.

Again, we combine this with injectivity on vertex groups to obtain a notion of an essential morphism.

Definition 1.11. A morphism of space pairs $f : (X, \underline{w}) \rightarrow (X', \underline{w}')$ is *essential* if it is ∂ -essential and $f : X \rightarrow X'$ is π_1 -injective.

Finally, we note that our two definitions of ∂ -essential pairs coincide.

Lemma 1.12. *Let $(X, \underline{w}) \rightarrow (X', \underline{w}')$ be a morphism of space pairs, inducing the corresponding morphism of pairs $(G, \mathcal{A}) \rightarrow (G', \mathcal{A}')$ on fundamental groups. The morphism $(X, \underline{w}) \rightarrow (X', \underline{w})$ is ∂ -essential if and only if the morphism $(G, \mathcal{A}) \rightarrow (G', \mathcal{A}')$ is ∂ -essential. Hence, $(X, \underline{w}) \rightarrow (X', \underline{w})$ is essential if and only if $(G, \mathcal{A}) \rightarrow (G', \mathcal{A}')$ is essential.*

Proof. The quotient of \tilde{X} by the action of $\ker f$ is \hat{X} , and the corresponding covering map induces a map $\tilde{Y} \rightarrow \hat{Y}$.

We need to show that two path components \tilde{Y}_1 and \tilde{Y}_2 of \tilde{Y} have the same image under this map if and only if they are in the same orbit of $\ker f$. The ‘if’ direction is clear. For the converse, we choose compatible basepoints $*_i \in \tilde{Y}_i$ and suppose that \tilde{Y}_1 and \tilde{Y}_2 have the same image. Then the images of their basepoints in \hat{X} are joined by a concatenation of paths $\kappa \cdot \eta$ where κ maps to a loop in \hat{X} and η is the image of a lift of a loop from Y to Y_2 . These define group elements $k \in \ker f$ and $y \in \text{Stab}_G(\tilde{Y}_2)$ so that gy translates \tilde{Y}_1 to \tilde{Y}_2 , and so \tilde{Y}_1 and \tilde{Y}_2 are indeed in the same orbit of $\ker f$.

The lemma follows immediately. \square

1.3 Graph pairs

In the setting of Theorem A, the groups G are finitely generated free groups, so the spaces X can be taken to be graphs. We may therefore apply the techniques of Stallings [38].

Definition 1.13. Let Γ be a graph. The *star* of a vertex v is the set $\text{St}_\Gamma(v) = \{e \in E \mid \iota(e) = v\}$, the set of edges with initial vertex v . (We will also write $\text{St}(v)$ for $\text{St}_\Gamma(v)$ when there is no fear of confusion.) A morphism of graphs $f : \Gamma \rightarrow \Delta$ is an *immersion* if the induced maps on stars are injective. In this case, we write $f : \Gamma \looparrowright \Delta$.

Stallings famously observed that immersions are π_1 -injective, and that any morphism of finite graphs $\Lambda \rightarrow \Gamma$ factors through a canonical immersion

$$\Lambda \rightarrow \Lambda_0 \looparrowright \Gamma$$

where the map $\Lambda \rightarrow \Lambda_0$ is a composition of finitely many folds [38, §§3.3].

Definition 1.14. A *multicycle* in a graph Γ is an immersion of graphs $\underline{w} : \mathbb{S}_{\underline{w}} \rightarrow \Gamma$, where $\mathbb{S}_{\underline{w}}$ is a disjoint union of graphs homeomorphic to circles. The components of $\mathbb{S}_{\underline{w}}$ are denoted by S_i^1 and the restriction of \underline{w} to S_i^1 is denoted by w_i .

A *graph pair* is a space pair (Γ, \underline{w}) , where Γ is a finite graph and \underline{w} is a multicycle. Note that we do not require the graph Γ to be connected.

Again, we will need a notion of morphism for graph pairs. As for Stallings, for us a morphism of graphs takes vertices to vertices and edges to edges. Since we insist that the maps \underline{w} are immersions, we make a corresponding requirement for morphisms of graph pairs.

Definition 1.15. Let (Γ, \underline{w}) and (Λ, \underline{u}) be graph pairs. A morphism of space pairs $(\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$ is a *morphism of graph pairs* if the map $\Lambda \rightarrow \Gamma$ is a morphism of graphs and the map $\mathbb{S}_{\underline{u}} \rightarrow \mathbb{S}_{\underline{w}}$ is an immersion.

The first advantage of this setting is that we can certify π_1 -injective maps using immersions. Note that a morphism of graphs $\Lambda \rightarrow \Gamma$ is an immersion if and only if the lift to universal covers $\tilde{\Lambda} \rightarrow \tilde{\Gamma}$ is injective. Similarly, we may define an immersion of graph pairs.

Definition 1.16. A map of graphs pairs $f : (\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$ is an *immersion* if the lifts $\tilde{\Lambda} \rightarrow \tilde{\Gamma}$ and $\tilde{\mathbb{S}}_{\underline{u}} \rightarrow \tilde{\mathbb{S}}_{\underline{w}}$ are injective. In this case, we write $f : (\Lambda, \underline{u}) \looparrowright (\Gamma, \underline{w})$.

A map of graph pairs factors through a canonical immersion, just as maps of graphs do.

Lemma 1.17. A map of graphs pairs $f : (\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$ factors through a canonical immersion $f_0 : (\Lambda_0, \underline{u}_0) \looparrowright (\Gamma, \underline{w})$. The immersion f_0 has the universal property that, whenever f factors through an immersion $(\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$, f_0 also factors through $(\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$.

Proof. Let G be the image of $\pi_1 \Lambda$ in $\pi_1 \Gamma$. We take $\tilde{\Lambda}_0$ to be the image of $\tilde{\Lambda}$ in $\tilde{\Gamma}$; likewise, we take $\tilde{\mathbb{S}}_{\underline{u}_0}$ to be the image of $\tilde{\mathbb{S}}_{\underline{u}}$ in $\tilde{\mathbb{S}}_{\underline{w}}$. The group G acts naturally on each of these, and we take Λ_0 and $\mathbb{S}_{\underline{u}_0}$ to be the respective quotients by the action of G . \square

The class of immersions is too rigid for our purposes, and we would prefer to deal with maps that are essential, but not necessarily immersions. Unfortunately, there does not seem to be a good way of certifying essential maps

‘locally’. However, the next lemma provides a means of locally certifying that a map is ∂ -essential.

Lemma 1.18. *If a map of graph pairs $f : (\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$ is ∂ -essential and $(\Lambda_0, \underline{u}_0) \looparrowright (\Gamma, \underline{w})$ is the corresponding canonical immersion then the induced map $\mathbb{S}_{\underline{u}} \rightarrow \mathbb{S}_{\underline{u}_0}$ is injective. Conversely, if a map of graph pairs $f : (\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$ factors through an immersion as*

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$$

and $\mathbb{S}_{\underline{u}} \rightarrow \mathbb{S}_{\underline{v}}$ is injective then f is ∂ -essential.

Proof. By definition, if f is ∂ -essential then the corresponding map $\widehat{\mathbb{S}}_{\underline{u}} \rightarrow \widetilde{\mathbb{S}}_{\underline{w}}$ is injective on π_0 . The components of these spaces are lines, so the map $\widehat{\mathbb{S}}_{\underline{u}} \rightarrow \widetilde{\mathbb{S}}_{\underline{w}}$ is itself injective, and so $\widehat{\mathbb{S}}_{\underline{u}} \rightarrow \widetilde{\mathbb{S}}_{\underline{u}_0}$ is injective too. Finally, since $\mathbb{S}_{\underline{u}} \rightarrow \mathbb{S}_{\underline{u}_0}$ is obtained by quotienting the domain and the range by $f_*\pi_1\Lambda$, it is also injective.

For the converse, if f factors as hypothesized, then the lift of $\mathbb{S}_{\underline{u}} \rightarrow \mathbb{S}_{\underline{w}}$ factors as

$$\widehat{\mathbb{S}}_{\underline{u}} \rightarrow \widetilde{\mathbb{S}}_{\underline{v}} \rightarrow \widetilde{\mathbb{S}}_{\underline{w}}.$$

The first map is a lift of an injection, hence an injection, and the second map is a lift of an immersion, hence injective. The result follows since a composition of injective maps is injective. \square

Thus, we can use a map to an immersed pair as a certificate that a morphism of pairs is ∂ -essential. We call the data of this certificate a ∂ -immersion.

Definition 1.19. A ∂ -immersion is a concatenation

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$$

where $\mathbb{S}_{\underline{u}} \rightarrow \mathbb{S}_{\underline{v}}$ is injective.

In general, a ∂ -essential morphism may be far from being essential. However, it turns out that we may promote ∂ -essential surfaces to essential surfaces – see Theorem 4.5 below.

2 Whitehead graphs and folds

Given a group pair $(F, \langle w \rangle)$, where w is some non-trivial element of a free group F , it is natural to ask whether or not $\langle w \rangle$ is a free factor of F . The standard way of answering this question uses the Whitehead graph, which was defined by J. H. C. Whitehead in his original paper on automorphisms of free groups [40]. (See also [14] and the references therein for a modern account of Whitehead graphs.) The definition of Whitehead graph given in [40] implicitly involves representing F as the fundamental group of a rose – a graph with a single vertex.

Here, we develop the theory of Whitehead graphs for general graph pairs. We are not aware that this approach has been taken in the literature before, but it is similar to the approaches to Whitehead graphs given by Cashen–Macura [14] and Manning [30].

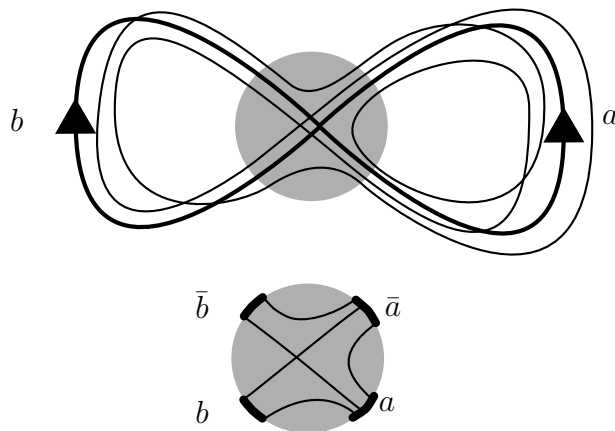


Figure 1: The Whitehead graph of the Baumslag–Solitar word $b^{-1}aba^{-2}$.

Definition 2.1. Consider a graph pair (Γ, \underline{w}) and a vertex x of Γ . The *Whitehead graph* at x is denoted by $\text{Wh}_x(\underline{w})$. Its set of vertices is the star $\text{St}_\Gamma(x)$. The unoriented edges of $\text{Wh}_x(\underline{w})$ are the vertices $\{x_i\}$ of $\mathbb{S}_{\underline{w}}$ that map to x ; the edge corresponding to x_i joins the vertices $\underline{w}(e_1)$ and $\underline{w}(e_2)$ of $\text{Wh}_x(\underline{w})$, where e_1 and e_2 are the two edges of $\mathbb{S}_{\underline{w}}$ with $\iota(e_j) = x_i$.

Note that the requirement that the multicyle \underline{w} is an immersion implies that the endpoints of any edge of $\text{Wh}_x(\underline{w})$ are distinct. However, each pair of vertices may be joined by many edges.

We can collect together all the Whitehead graphs at the vertices of Γ into a global Whitehead graph for the pair (Γ, \underline{w}) .

Definition 2.2. The *Whitehead graph* of the pair (Γ, \underline{w}) is the disjoint union

$$\text{Wh}(\underline{w}) := \coprod_{x \in V} \text{Wh}_x(\underline{w}) .$$

Note that $\text{Wh}(\underline{w})$ comes equipped with two additional structures:

- (i) the components of $\text{Wh}(\underline{w})$ are naturally partitioned: two components are equivalent if they are both components of some $\text{Wh}_x(\underline{w})$;
- (ii) the fixed-point free involution $e \mapsto \bar{e}$ on the edges of Γ defines a fixed-point free involution on the vertices of $\text{Wh}(\underline{w})$ that extends to a bijection $i_e : \text{St}(e) \rightarrow \text{St}(\bar{e})$.

We will always think of $\text{Wh}(\underline{w})$ as equipped with these extra structures.

Remark 2.3. The partition on the components of $\text{Wh}_x(\underline{w})$ and the involutions i_e are enough information to reconstruct the pair (Γ, \underline{w}) .

Stallings studied morphisms of graphs by observing that they always factor as a composition of a sequence of *folds* followed by an immersion. Recall that a fold identifies a pair of edges e_1, e_2 with $\iota(e_1) = \iota(e_2)$. It is therefore natural to study the effect that a fold has on Whitehead graphs.

Definition 2.4. Let W be a graph and v_1, v_2 a pair of vertices. (When we apply this, W will be a disjoint union of Whitehead graphs.) A *wedge* is a quotient map $W \rightarrow W'$ that identifies v_1 and v_2 and leaves the rest of W unchanged. We write $W \wedge_{v_1 \sim v_2}$ for the quotient graph W' . If $W = W_1 \sqcup W_2$ with $v_i \in W_i$ then we write $W_1 \wedge_{v_1 \sim v_2} W_2$ for $W \wedge_{v_1 \sim v_2}$. The reverse move, which replaces $W \wedge_{v_1 \sim v_2}$ by W , is called an *unwedge*.

The following lemma shows that, at the level of Whitehead graphs, folds correspond to wedges.

Lemma 2.5. *Let $f : (\Gamma, \underline{w}) \rightarrow (\Gamma', \underline{w}')$ be the morphism given by a fold $\Gamma \rightarrow \Gamma'$, which identifies two edges e_1, e_2 of Γ with a common initial vertex $x = \iota(e_1) = \iota(e_2)$ to an edge e' of Γ' . Let $y_i = \iota(\bar{e}_i)$ and let $x' = \iota(e')$ and*

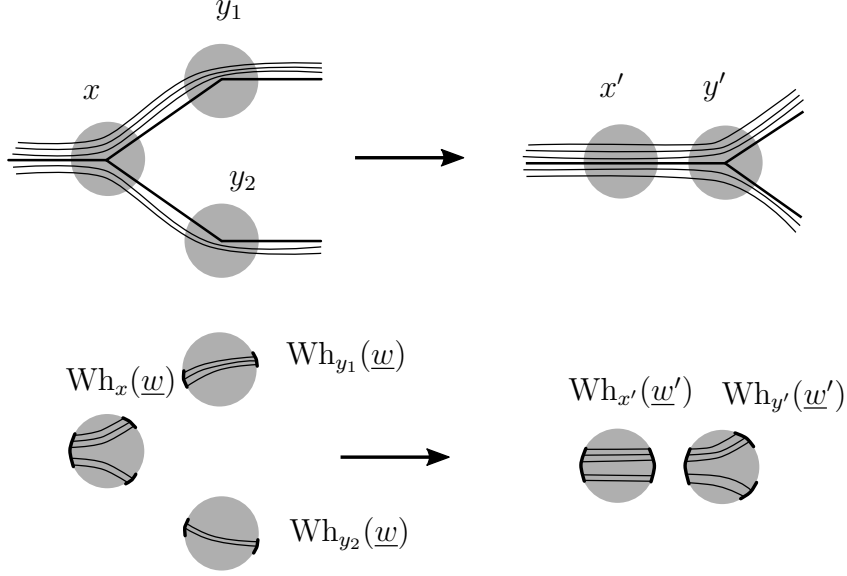


Figure 2: The effect of a fold on Whitehead graphs. Note that $\text{Wh}_{x'}(\underline{w}')$ is obtained by wedging $\text{Wh}_x(\underline{w})$, and $\text{Wh}_{y'}(\underline{w}')$ is obtained by wedging $\text{Wh}_{y_1}(\underline{w}) \sqcup \text{Wh}_{y_2}(\underline{w})$.

$y' = \tau(e')$. Suppose further that $f : \Gamma \rightarrow \Gamma'$ is a homotopy equivalence, i.e. $y_1 \neq y_2$. Then

$$\text{Wh}_{x'}(\underline{w}') \cup \text{Wh}_{y'}(\underline{w}') = (\text{Wh}_x(\underline{w}) \wedge_{e_1 \sim e_2}) \cup (\text{Wh}_{y_1}(\underline{w}) \wedge_{\bar{e}_1 \sim \bar{e}_2} \text{Wh}_{y_2}(\underline{w})) .$$

(Note that the unions in this expression may not be disjoint, since x may equal y_i for at most one i , in which case x' also equals y'). In particular, \bar{e}' is a cut vertex of $\text{Wh}_{y'}(\underline{w}')$.

Proof. This follows immediately from the definitions. (The case in which $x \neq y_1, y_2$ is illustrated in Figure 2). \square

Remark 2.6. In the above lemma, the hypothesis that the map $f : (\Gamma, \underline{w}) \rightarrow (\Gamma', \underline{w}')$ is a *morphism of pairs* is essential: if the induced map $\mathbb{S}_{\underline{w}} \rightarrow \mathbb{S}_{\underline{w}'}$ were not an immersion, then after folding one would need to tighten \underline{w}' to an immersion, which might destroy the cut point structure of the Whitehead graphs.

Remark 2.7. In the setting of Lemma 2.5, for any vertex v of Γ , the map $\text{Wh}_v(\underline{w}) \rightarrow \text{Wh}_{f(v)}(\underline{w}')$ induced by f is injective on edges.

In fact, the implication of Lemma 2.5 can be reversed: if one of the Whitehead graphs has a cut vertex then we can unfold.

Lemma 2.8. *Let $(\Gamma', \underline{w}')$ be a graph pair, and suppose that some edge \bar{e}' defines a cut vertex in $\text{Wh}_{y'}(\underline{w}')$ (where $y' = \tau(e')$). Then there is a graph pair (Γ, \underline{w}) and a morphism of pairs defining a homotopy-equivalent fold $f : (\Gamma, \underline{w}) \rightarrow (\Gamma', \underline{w}')$ that identifies a pair of edges e_1, e_2 to e' .*

Proof. The hypothesis tells us that $\text{Wh}_{y'}(\underline{w}') = W_1 \wedge_{\bar{e}_1 \sim \bar{e}_2} W_2$, where $\bar{e}_i \in W_i$ is a vertex with image \bar{e}' in the wedge. Let $x' = \iota(e')$ (and note that x' and y' are not necessarily distinct). We will define the pair (Γ, \underline{w}) via its Whitehead graphs (appealing to Remark 2.3). The proof divides into two similar cases, depending on whether or not $x' = y'$.

Suppose first that $x' \neq y'$. For any vertex z' of Γ' not equal to x' or y' , we take a vertex z for Γ with Whitehead graph isomorphic to $\text{Wh}_{z'}(\underline{w}')$. The remaining vertices of Γ are denoted by x, y_1, y_2 . We define $\text{Wh}_{y_i}(\underline{w})$ to be W_i for $i = 1, 2$. Finally, $\text{Wh}_x(\underline{w})$ is defined so that

$$\text{Wh}_{x'}(\underline{w}') = \text{Wh}_x(\underline{w}) \wedge_{e_1 \sim e_2} .$$

That is, $\text{Wh}_x(\underline{w})$ is obtained from $\text{Wh}_{x'}(\underline{w}')$ by dividing the vertex e' into two vertices, e_1, e_2 . The edges of $\text{Wh}_x(\underline{w})$ incident at e_1 and e_2 are defined so that they respect the natural bijections between the stars of the e_i and the stars of the \bar{e}_i . There is then a natural lift of the bijections on stars in $\text{Wh}(\underline{w}')$ to bijections on stars in $\text{Wh}(\underline{w})$, and this completes the construction of (Γ, \underline{w}) .

The case in which $x' = y'$ is similar. Again, for any vertex z' of Γ' not equal to x' or y' , we take a vertex z for Γ with Whitehead graph isomorphic to $\text{Wh}_{z'}(\underline{w}')$. The remaining vertices of Γ are denoted by x, y . Since $x' = y'$, the vertex e' is contained in $\text{Wh}_{y'}(\underline{w}') = W_1 \wedge_{\bar{e}_1 \sim \bar{e}_2} W_2$, and without loss of generality we may take $e' \in W_1$. We now define $\text{Wh}_y(\underline{w}) = W_2$, and $\text{Wh}_x(\underline{w})$ so that

$$W_1 = \text{Wh}_x(\underline{w}) \wedge_{e_1 \sim e_2} .$$

As before, this means that $\text{Wh}_x(\underline{w})$ is obtained by dividing the vertex e' into two vertices, e_1, e_2 , and the edges of $\text{Wh}_x(\underline{w})$ incident at the e_i are defined to respect the natural bijections between the stars of the e_i and the stars of the \bar{e}_i . Again, this completes the construction of (Γ, \underline{w}) .

In either case, identifying e_1 and e_2 defines a fold $(\Gamma, \underline{w}) \rightarrow (\Gamma', \underline{w}')$. Since $\tau(e_2) \neq \tau(e_1)$, the fold is a homotopy equivalence. Note also that, by construction, the fold is a morphism of pairs. \square

Whitehead introduced Whitehead graphs to recognize basis elements of free groups and, more generally, free splittings. (More generally still, Whitehead gave an algorithm to find the shortest element in an orbit of the automorphism group.) We will use Whitehead graphs to recognize *irreducible* pairs.

Definition 2.9. We will call the pair (Γ, \underline{w}) *reducible* if: either

- (i) $\pi_1\Gamma$ is cyclic, $\mathbb{S}_{\underline{w}}$ has one component and \underline{w} is a π_1 -isomorphism; or
- (ii) there is a non-trivial free splitting of $\pi_1\Gamma$ relative to the family of cyclic subgroups $\{w_{i*}\pi_1 S_i^1\}$.

If neither of these conditions hold then we call the pair (Γ, \underline{w}) *irreducible*.

The point of the above definition is that the pair (Γ, \underline{w}) is reducible if and only if the fundamental group of the double $D(\underline{w})$, obtained as a graph of spaces with two vertex spaces homeomorphic to Γ and edge maps given by \underline{w} , admits a non-trivial free splitting. Item (i) arises because in this instance the fundamental group of the double is \mathbb{Z} , which splits as an HNN extension of the trivial group.

The following lemma is the key result for recognizing free splittings. It is quite standard, but we give a proof using folds and wedges as a sample application of the above ideas.

Lemma 2.10 (Whitehead). *Suppose that \underline{w} is an immersion. If (Γ, \underline{w}) is reducible then there is a vertex x of Γ so that one of the following holds:*

- (i) $\text{Wh}_x(\underline{w})$ is disconnected;
- (ii) $\text{Wh}_x(\underline{w})$ has a leaf, i.e. a vertex of valence 1; or
- (iii) $\text{Wh}_x(\underline{w})$ has a cut vertex, i.e. a vertex e so that $\text{Wh}_x(\underline{w}) \setminus \{e\}$ is disconnected.

Proof. The case in which $\pi_1\Gamma$ is cyclic and $\underline{w} : S^1 \rightarrow \Gamma$ is a π_1 -isomorphism is easy and left as an exercise. Suppose therefore that $\pi_1\Gamma$ admits a free splitting relative to \underline{w} . It follows that there is a morphism of graphs $f :$

$(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w})$ which is a homotopy equivalence, so that Γ' has a vertex x' with $\text{Wh}_{x'}(\underline{w}')$ disconnected. The morphism f now factors as a sequence of homotopy-equivalent folds; in particular, whenever e_1 and e_2 with $\iota(e_1) = \iota(e_2)$ are identified, we have $\tau(e_1) \neq \tau(e_2)$. Consider the final such fold, which identifies a pair of distinct vertices y_1, y_2 to a vertex y . Lemma 2.5 implies that $\text{Wh}_y(\underline{w})$ has a cut vertex. \square

Motivated by Whitehead's lemma, we call a Whitehead graph $\text{Wh}_x(\underline{w})$ *reducible* if it satisfies one of the three conclusions of the lemma; otherwise, we call $\text{Wh}_x(\underline{w})$ *irreducible*. We call the pair (Γ, \underline{w}) *locally irreducible* if, for every vertex x of Γ , the Whitehead graph $\text{Wh}_x(\underline{w})$ is irreducible. Whitehead's lemma therefore says that a locally irreducible pair is irreducible.

The converse to this statement is not quite true – there are irreducible pairs that are not locally irreducible. To construct an example, take an irreducible pair and apply one fold. We therefore must allow ourselves to unfold in order to prove a converse to Whitehead's lemma.

Lemma 2.11 (Converse to Whitehead's lemma). *If a pair (Γ, \underline{w}) is irreducible then there is a locally irreducible pair $(\Gamma', \underline{w}')$ and a map of pairs $(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w})$ which is a homotopy equivalence.*

Proof. If some vertex of Γ has either a disconnected Whitehead graph or a leaf then (Γ, \underline{w}) is reducible. Suppose therefore that there is a vertex y so that $\text{Wh}_y(\underline{w})$ has a cut vertex \bar{e} . By Lemma 2.8, there is a fold $(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w})$ so that the edge e is unfolded to a pair of edges e'_1, e'_2 .

Note that: the number of edges of $\text{Wh}(\underline{w}')$ is equal to the number of edges of $\text{Wh}(\underline{w})$; the number of vertices of $\text{Wh}(\underline{w}')$ is greater than the number of vertices of $\text{Wh}(\underline{w})$; for each vertex z' of Γ' , the Whitehead graph $\text{Wh}_{z'}(\underline{w}')$ is connected without leaves. In particular, the number of vertices of $\text{Wh}(\underline{w}')$ is at most the number of edges of $\text{Wh}(\underline{w})$. It follows that only finitely many unfoldings of this form can be performed.

When no further unfoldings can be performed, the final pair $(\Gamma', \underline{w}')$ is locally irreducible, as claimed. \square

3 Admissible ∂ -immersions

Consider an irreducible graph pair (Γ, \underline{w}) as above. In this section, we will study ∂ -immersions

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$$

where (Λ, \underline{u}) is a locally irreducible pair. We will impose one additional condition on our ∂ -immersions.

Definition 3.1. A map of graph pairs $(\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$ is called *admissible* if there is a positive integer $n = n(\underline{u})$ so that every point in $\mathbb{S}_{\underline{w}}$ has exactly n preimages in $\mathbb{S}_{\underline{u}}$. A ∂ -immersion $(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$ is called *admissible* if the composition $(\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$ is admissible.

Calegari uses the term ‘admissible’ similarly for maps of surfaces with boundary [8, p. 37]. Note that, in his context, the integer n counts preimages with a sign determined by orientation, whereas in our context, the count is unsigned.

We next write down a finite set of pieces, from which ∂ -immersions of locally irreducible pairs can be constructed.

Definition 3.2. Let \mathcal{W} be the set of components of $\text{Wh}(\underline{w})$. The set \mathcal{P} of *pieces* (over $\text{Wh}(\underline{w})$) consists of all pairs of maps of graphs (up to graph isomorphism)

$$P \rightarrow V \hookrightarrow W$$

such that P is a disjoint union of irreducible graphs, the map $P \rightarrow V$ is bijective on edges, the map $V \hookrightarrow W$ is injective, and $W \in \mathcal{W}$. When $P \rightarrow V \hookrightarrow W$ is an element of \mathcal{P} , we will often abuse notation and write $P \in \mathcal{P}$, since the map $P \rightarrow W$ determines V .

Remark 3.3. Note that \mathcal{P} is finite. This trivial observation is of crucial importance.

We will study admissible ∂ -immersions of locally irreducible pairs by looking at how they are constructed from the pieces \mathcal{P} . The involutions i_e on the stars of vertices of $\text{Wh}(\underline{w})$ define relations on the elements of \mathcal{P} , as follows.

Definition 3.4. Consider

$$P \rightarrow U \hookrightarrow W_1, \quad Q \rightarrow V \hookrightarrow W_2$$

elements of \mathcal{P} . Suppose that e is a vertex of $U \subseteq W_1$ and that \bar{e} is a vertex of $V \subseteq W_2$. Let e_1, \dots, e_m be the set of vertices of P that map to $e \in U$, and let $\bar{e}_1, \dots, \bar{e}_n$ be the set of vertices of Q that map to $\bar{e} \in V$. We write

$$P \leftrightarrow_e Q$$

if:

- (i) $m = n$, and
- (ii) up to reordering of indices, i_e restricts to bijections $\text{St}_P(e_j) \rightarrow \text{St}_Q(\bar{e}_j)$ for all j .

The relation $P \leftrightarrow_e Q$ can be interpreted in terms of Manning's *splicing* operation [30]. It says that P and Q can be *spliced* at the sets of vertices $\{e_1, \dots, e_n\}$ and $\{\bar{e}_1, \dots, \bar{e}_n\}$, and that U and V can be spliced at e and \bar{e} .

To record how the elements of \mathcal{P} are glued together, we introduce \mathcal{P} -stars.

Definition 3.5. A \mathcal{P} -star σ consists of the following data:

- (i) a piece $P \rightarrow U \hookrightarrow W$ in \mathcal{P} ;
- (ii) for each vertex e_i of U , a choice of piece $Q_i \rightarrow V_i \hookrightarrow W_i$ in \mathcal{P} and a vertex \bar{e}_i of V_i so that $P \leftrightarrow_{e_i} Q_i$.

We write $\sigma(*) = P$ and, for each vertex e_i of P , we write $\sigma(e_i) = Q_i$.

Let $\mathcal{S} \equiv \mathcal{S}(\mathcal{P})$ be the (finite) set of all \mathcal{P} -stars. Let $V_{\mathcal{P}} = \mathbb{R}^{\mathcal{S}}$ and let $V_{\mathcal{P}}^+ \subseteq V_{\mathcal{P}}$ be the non-negative orthant. An admissible ∂ -immersion

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$$

of a locally irreducible pair (Λ, \underline{u}) defines an integer vector $\pi(\underline{u}) \in V_{\mathcal{P}}^+$ in a natural way, as follows. For each vertex x of Δ , let x'_1, \dots, x'_k be the preimages of x in Λ and let x_0 be the image vertex in Γ . The piece $P(x) \in \mathcal{P}$ is then defined to be

$$\prod_{i=1}^k \text{Wh}_{x'_i}(\underline{u}) \rightarrow \text{Wh}_x(\underline{v}) \hookrightarrow \text{Wh}_{x_0}(\underline{w}) .$$

If e is an edge of Δ with $\iota(e) = x$ and $\tau(e) = y$, then $P(x) \leftrightarrow_e P(y)$. Therefore, for each vertex x of Δ , we can define σ_x to be the corresponding \mathcal{P} -star associated to the labels of the neighbouring vertices:

- (i) $\sigma_x(*) = P(x)$; and,
- (ii) for each $e_i \in \text{St}_{\Delta}(x)$, $\sigma_x(e_i) = P(\tau(e_i))$.

Now define $\pi(\underline{u})$ to be the vector $\underline{p} \in V_{\mathcal{P}}^+$ so that

$$p_{\sigma} = \#\{x \in V(\Delta) \mid \sigma_x = \sigma\}$$

for each $\sigma \in \mathcal{S}$.

The image of π is not arbitrary: in fact, it is precisely the set of integer points of a certain cone $C_{\mathcal{P}} \subseteq V_{\mathcal{P}}^+$. We will describe this cone using systems of equations: the gluing equations and the admissibility equations. We start with the gluing equations. A non-negative integer vector that satisfies the gluing equations necessarily comes from a ∂ -immersion of a locally irreducible pair.

Definition 3.6. We write $\underline{x} = (x_{\sigma})_{\sigma \in \mathcal{S}}$ for an element of $V_{\mathcal{P}}$. For each pair of pieces $P, Q \in \mathcal{P}$ and edge e satisfying $P \leftrightarrow_e Q$ we have the *gluing equation*

$$\sum_{\substack{\sigma(*)=P \\ \sigma(e)=Q}} x_{\sigma} = \sum_{\substack{\sigma(*)=Q \\ \sigma(\bar{e})=P}} x_{\sigma}$$

where each sum is taken over all \mathcal{P} -stars σ satisfying the conditions.

We next describe the admissibility equations, which force any \mathcal{P} -pair that defines a vector to be admissible.

Definition 3.7. Let ϵ be an edge of $\text{Wh}(\underline{w})$. For a piece

$$P \rightarrow V \hookrightarrow W$$

in \mathcal{P} , set $\delta_{\epsilon}(P)$ to be the number of preimages of ϵ in P . (Note that this is either 0 or 1, by definition.) We now define a linear map $n_{\epsilon} : V_{\mathcal{P}} \rightarrow \mathbb{R}$ by setting

$$n_{\epsilon}(\underline{x}) = \sum_{\sigma \in \mathcal{S}} x_{\sigma} \delta_{\epsilon}(\sigma(*)) .$$

The *admissibility equations* assert that $n_{\epsilon}(\underline{x}) = n_{\epsilon'}(\underline{x})$ for all edges ϵ and ϵ' in $\text{Wh}(\underline{w})$.

The cone $C_{\mathcal{P}} \subseteq V_{\mathcal{P}}^+$ is now defined to be the subset of $V_{\mathcal{P}}^+$ that satisfies the gluing equations and the admissibility equations.

Lemma 3.8. *An integer vector $\underline{x} \in V_{\mathcal{P}}^+ \setminus 0$ is the image of an admissible ∂ -immersion from a locally irreducible pair under π if and only if it is in $C_{\mathcal{P}}$.*

Proof. Let (Λ, \underline{u}) be locally irreducible and

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$$

a ∂ -immersion. First we show that $\pi(\underline{u})$ satisfies the gluing equations. Indeed, the expression in the gluing equations is just two different ways of evaluating the number of edges e of Δ with $P(\iota(e)) = P$ and $P(\tau(e)) = Q$. The admissibility equations are satisfied since each n_e evaluates to $n(\underline{u})$.

Conversely, given an integer vector $\underline{x} \in C_{\mathcal{P}}$, we need to construct an admissible ∂ -immersion

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$$

with (Λ, \underline{u}) locally irreducible. By Remark 2.3, it is enough to describe

$$\text{Wh}(\underline{u}) \rightarrow \text{Wh}(\underline{v}) \hookrightarrow \text{Wh}(\underline{w})$$

together with their pairings on stars of vertices. For each star σ , $\text{Wh}(\underline{u}) \rightarrow \text{Wh}(\underline{v}) \hookrightarrow \text{Wh}(\underline{w})$ contains x_σ copies of the piece $\sigma(*)$. This determines the graphs and maps $\text{Wh}(\underline{u}) \rightarrow \text{Wh}(\underline{v}) \hookrightarrow \text{Wh}(\underline{w})$; it remains to determine the pairings. Consider the pieces

$$P \rightarrow U \hookrightarrow W_1, \quad Q \rightarrow V \hookrightarrow W_2$$

and suppose that e is an edge of U and \bar{e} is an edge of V . The gluing equations imply that here is a bijection between the number of \mathcal{P} -stars σ so that $\sigma(*) = P$ and $\sigma(e) = Q$ and the number of \mathcal{P} -stars σ so that $\sigma(*) = Q$ and $\sigma(\bar{e}) = P$ which satisfy the condition that the bijection $i_e : \text{St}_{W_1}(e) \rightarrow \text{St}_{W_2}(\bar{e})$ restricts to a bijection $\text{St}_U(e) \rightarrow \text{St}_V(\bar{e})$, and thence to bijections of the stars of the preimages in P and Q . Choosing a bijection between these \mathcal{P} -stars then determines the required bijection between vertices of the copies of P and Q in these \mathcal{P} -stars, and the bijections between stars are then determined by the relation \leftrightarrow_e .

By construction, (Λ, \underline{u}) is locally irreducible and

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$$

is a ∂ -immersion. Finally, the admissibility equations immediately imply that this ∂ -immersion is admissible. \square

Thus, we have seen that admissible ∂ -immersions of locally irreducible pairs correspond naturally to non-zero integer vectors in $C_{\mathcal{P}}$ or, equivalently, to rational points in the projectivization $\mathbb{P}(C_{\mathcal{P}})$. Motivated by Calegari's work on stable commutator length (see [9] and [8], and also [5]), we will study these rational points via rational functions on $\mathbb{P}(C_{\mathcal{P}})$.

4 The rationality theorem

We start by writing down two natural linear maps on $C_{\mathcal{P}}$. For an admissible ∂ -immersion

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w}) ,$$

the corresponding linear maps are (minus) the Euler characteristic of Λ , and the degree $n(\underline{u})$ with which $\mathbb{S}_{\underline{u}}$ covers $\mathbb{S}_{\underline{w}}$. The key observation is that both of these can be computed from the vector $\pi(\underline{u})$.

First, the admissibility equations imply that the linear map n_{ϵ} is independent of ϵ . We therefore write $n = n_{\epsilon}$, evidently a linear map which is non-zero on $C_{\mathcal{P}} \setminus 0$.

Second, for a piece

$$P \rightarrow V \hookrightarrow W$$

in \mathcal{P} , we let $\mu(P)$ denote the number of connected components of P , and let $\nu(P)$ denotes the number of vertices of P . We then define $\chi_- : V_{\mathcal{P}} \rightarrow \mathbb{R}$ by

$$\chi_-(\underline{x}) = \sum_{\sigma \in \mathcal{S}} x_{\sigma} \left(\frac{1}{2} \nu(\sigma(*)) - \mu(\sigma(*)) \right) .$$

It's well known that the Euler characteristic of a graph can be computed as the sum over the vertices of one minus half the valence, and from this we see that, for an admissible ∂ -immersion

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$$

of a locally irreducible pair (Λ, \underline{u}) , we have $\chi_- \circ \pi(\underline{u}) = -\chi(\Lambda)$.

Definition 4.1. Since χ_-/n is a quotient of two linear maps on $C_{\mathcal{P}} \setminus 0$ and the denominator is non-zero, it yields a well defined function on the projectivization $\mathbb{P}(C_{\mathcal{P}})$. We call this function

$$\rho_{\mathcal{P}} = \frac{\chi_-}{n}$$

the *projective \mathcal{P} -rank* function on $\mathbb{P}(C_{\mathcal{P}})$.

Remark 4.2. Suppose $(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$ is an admissible ∂ -immersion of a locally irreducible pair. Suppose that $(\Lambda', \underline{u}') \rightarrow (\Lambda, \underline{u})$ is a d -sheeted covering map. Then $(\Lambda', \underline{u}') \rightarrow (\Gamma, \underline{w})$ is ∂ -essential and, therefore, by Lemma 1.18,

$$(\Lambda', \underline{u}') \rightarrow (\Lambda'_0, \underline{u}'_0) \looparrowright (\Gamma, \underline{w})$$

is also a ∂ -immersion of a locally irreducible pair. Furthermore, since $\chi(\Lambda') = d \cdot \chi(\Lambda)$ and $n(\underline{u}') = d \cdot n(\underline{u})$, we see that $\rho_{\mathcal{P}} \circ \pi(\underline{u}') = \rho_{\mathcal{P}} \circ \pi(\underline{u})$.

In analogy with stable commutator length (see [8]), we may use the rational function $\rho_{\mathcal{P}}$ to define an invariant of a multicycle \underline{w} in a graph Γ .

Definition 4.3. The *maximal \mathcal{P} -rank* of a pair (Γ, \underline{w}) is denoted by $\rho_{\mathcal{P}}^+(\underline{w})$ and defined to be

$$\rho_{\mathcal{P}}^+(\underline{w}) := \max_{[\underline{x}] \in \mathbb{P}(C_{\mathcal{P}})} \rho_{\mathcal{P}}[\underline{x}] .$$

Note that this maximum is indeed realized, since $\mathbb{P}(C_{\mathcal{P}})$ is compact. Similarly, the *minimal \mathcal{P} -rank*, $\rho_{\mathcal{P}}^-(\underline{w})$, is defined to be the minimum of $\rho_{\mathcal{P}}$ over the same domain.

Since $\mathbb{P}(C_{\mathcal{P}})$ is compact, the maximal and minimal \mathcal{P} -ranks are certainly attained as long as $\mathbb{P}(C_{\mathcal{P}})$ is non-empty (i.e. as long as $C_{\mathcal{P}}$ is non-zero). In fact, since $\rho_{\mathcal{P}}$ is a quotient of linear maps, the maximal and minimal \mathcal{P} -ranks are attained on rational points of $\mathbb{P}(C_{\mathcal{P}})$, and hence are realized by admissible ∂ -immersions.

Theorem 4.4. *If $C_{\mathcal{P}} \neq 0$, then the maximal and minimal \mathcal{P} -ranks are realized by admissible ∂ -immersions of locally irreducible pairs; that is, there exist locally irreducible pairs $(\Lambda_{\pm}, \underline{u}_{\pm})$ and admissible ∂ -immersions*

$$(\Lambda_{\pm}, \underline{u}_{\pm}) \rightarrow (\Delta_{\pm}, \underline{v}_{\pm}) \looparrowright (\Gamma, \underline{w})$$

so that $\rho_{\mathcal{P}} \circ \pi(\underline{u}_{\pm}) = \rho_{\mathcal{P}}^{\pm}(\underline{w})$. In particular, $\rho_{\mathcal{P}}^{\pm}(\underline{w})$ are positive rational numbers.

Proof. We prove the result for the maximal \mathcal{P} -rank; the proof for the minimal \mathcal{P} -rank is identical. If $C_{\mathcal{P}} \neq 0$ then the projectivization $\mathbb{P}(C_{\mathcal{P}})$ is non-empty. From the definition of $\rho_{\mathcal{P}}^+$, we may normalize and restrict our attention to the rational polytope $n^{-1}(1)$, so

$$\rho_{\mathcal{P}}^+(\underline{w}) = \max_{n(\underline{x})=1} \chi_{-}(\underline{x}) .$$

But χ_{-} is linear, and so attains its maximum on a vertex \underline{x}_0 of $n^{-1}(1)$. Since \underline{x}_0 is rational and $\rho_{\mathcal{P}}^+$ is a projective function, there is some integer vector \underline{x}_1 , a multiple of \underline{x}_0 , on which $\rho_{\mathcal{P}}^+$ attains its maximum. Since $\underline{x}_1 \in C_{\mathcal{P}}$ is an integer vector, it is equal to $\pi(\underline{u})$ for some admissible \mathcal{P} -pair (Λ, \underline{u}) , which therefore realizes $\rho_{\mathcal{P}}^+$, as required. \square

An admissible ∂ -immersion

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w}) ,$$

of a locally irreducible pair (Λ, \underline{u}) for which $\rho_{\mathcal{P}} \circ \pi(\underline{u}) = \rho_{\mathcal{P}}^+(\underline{w})$ is called *maximal*. (Similarly, if $\rho_{\mathcal{P}} \circ \pi(\underline{u}) = \rho_{\mathcal{P}}^-(\underline{w})$ then the ∂ -immersion is called *minimal*.)

The above theorem is very much in the spirit of Calegari's proof of the rationality of stable commutator length in free groups [9], modulo certain additional questions of orientability. It is possible to give a proof of the main theorem of [9] along very similar lines.

The following consequence of the results of [9] will enable us to promote ∂ -essential surfaces to essential surfaces. The reader is also referred to [8] for a comprehensive account of stable commutator length and extremal surfaces in free groups.

Theorem 4.5. *Suppose that (Γ, \underline{w}) is a graph pair. If there exists a compact surface with boundary Σ and a ∂ -essential, admissible map $(\Sigma, \partial\Sigma) \rightarrow (\Gamma, \underline{w})$, then there exists a compact surface with boundary Σ_0 and an essential, admissible map $(\Sigma_0, \partial\Sigma_0) \rightarrow (\Gamma, \underline{w})$.*

Proof. Passing to a double cover if necessary, we may assume that $(\Sigma, \partial\Sigma)$ is oriented. By a result of Culler [15], we may assume that $(\Sigma, \partial\Sigma) \rightarrow (\Gamma, \underline{w})$ is realized by a map of graph pairs (where $(\Sigma, \partial\Sigma)$ is realized by a fatgraph, i.e. a graph pair in which the Whitehead graph at every vertex is a circle). Let $(\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$ be the result of folding the map $(\Sigma, \partial\Sigma) \rightarrow (\Gamma, \underline{w})$. By Lemma 1.18, the map $\partial\Sigma \rightarrow \mathbb{S}_{\underline{v}}$ is a homeomorphism. Therefore, we may equip $\mathbb{S}_{\underline{v}}$ with the orientation of $\partial\Sigma$. In this case, \underline{v} now defines a boundary cycle in Δ , and so by the main theorem of [9], some integer multiple of \underline{v} bounds an extremal surface $(\Sigma_0, \partial\Sigma_0) \rightarrow (\Delta, \underline{v})$. But extremal surfaces are essential: [9, Lemma 2.9] shows that extremal surfaces are π_1 -injective, and a similar argument using boundary compressions shows that they are also ∂ -essential. Since $(\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$ is also essential and admissible, and a composition of essential, admissible maps is essential and admissible, this completes the proof. \square

5 Maximal \mathcal{P} -rank and surfaces

Our results so far imply that every irreducible pair admits a maximal ∂ -immersion.

Lemma 5.1. *If (Γ, \underline{w}) is irreducible then there exists a maximal, admissible ∂ -immersion*

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{u}) \looparrowright (\Gamma, \underline{w})$$

for a locally irreducible pair (Λ, \underline{u}) .

Proof. Since (Γ, \underline{w}) is irreducible, Lemma 2.11 guarantees a locally irreducible pair $(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w})$. The map $(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w})$ consists of a π_1 -isomorphism and a homeomorphism $\mathbb{S}_{\underline{w}'} \rightarrow \mathbb{S}_{\underline{w}}$, so is certainly admissible and essential. In particular,

$$(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w}) \xrightarrow{\cong} (\Gamma, \underline{w})$$

is an admissible ∂ -immersion of a locally irreducible pair, so $\pi(\underline{w}') \in C_{\mathcal{P}}$ and $C_{\mathcal{P}} \neq 0$. Theorem 4.4 now implies that a maximal ∂ -immersion exists. \square

In this section, we shall use the relative JSJ decomposition together with the results of [42] to show that maximal ∂ -immersions are closely related to surfaces.

Definition 5.2. Let (Γ, \underline{w}) be a graph pair, and consider the mapping cylinder $M_{\underline{w}}$ of $\underline{w} : \mathbb{S} \rightarrow \Gamma$. The pair (Γ, \underline{w}) is said to be of *surface type* if the pair of spaces $(M_{\underline{w}}, \mathbb{S}_{\underline{w}})$ is homotopy equivalent to a pair $(\Sigma, \partial\Sigma)$, where Σ is a compact surface with boundary.

A theorem of Culler [15] shows that any pair (Γ, \underline{w}) of surface type can be unfolded to a *fatgraph* $(\Gamma', \underline{w}')$ – a graph pair in which every Whitehead graph is a cycle. One may therefore equivalently think of pairs of surface type as given by fatgraphs.

Fundamental groups of pairs of surface type can typically be decomposed as graphs of groups in many ways. In order to discuss this, we introduce some terminology for graph-of-groups decompositions of pairs.

Definition 5.3. Let (G, \mathcal{A}) be a group pair. A *decomposition* of (G, \mathcal{A}) is a graph of groups \mathcal{G} with fundamental group G such that, for every $a \in \mathcal{A}$, the stabilizer $\text{Stab}_G(a)$ is conjugate into a vertex group of \mathcal{G} .

Let v be a vertex of \mathcal{G} , and fix a pre-image \tilde{v} of v in the Bass–Serre tree T . Let $G_{\tilde{v}}$ be the stabilizer of \tilde{v} . Set

$$\mathcal{A}_{\tilde{v}} = \{a \in \mathcal{A} \mid \text{Stab}_{G_{\tilde{v}}}(a) \neq 1\}$$

and let $\mathcal{B}_{\tilde{v}}$ be the set of edges of T incident at \tilde{v} . The *induced pair* at v is defined to be $(G_{\tilde{v}}, \mathcal{A}_{\tilde{v}} \sqcup \mathcal{B}_{\tilde{v}})$, which is defined up to conjugacy in G . The vertex v is called *peripheral* if $\mathcal{A}_{\tilde{v}}$ is non-empty.

If every edge group of \mathcal{G} is cyclic then \mathcal{G} is said to be a *cyclic decomposition* of (G, \mathcal{A}) . As usual, the graph of groups \mathcal{G} is called *trivial* if G is the stabilizer of some vertex of the Bass–Serre tree.

We will only be concerned with cyclic decompositions of graph pairs (Γ, \underline{w}) , with $F = \pi_1\Gamma$. We will abuse notation and write the corresponding group pair as (F, \underline{w}) .

Pairs of surface type can be contrasted with rigid pairs, which only have trivial decompositions.

Definition 5.4. A graph pair (Γ, \underline{w}) is *rigid* if every cyclic decomposition of (Γ, \underline{w}) is trivial, and if (Γ, \underline{w}) is not of surface type. (This last requirement is to rule out the pair of pants, which is of surface type but admits no cyclic decompositions.) A group pair (F, \underline{w}) is *rigid* if some (any) corresponding graph pair is rigid.

The main theorem of this section is phrased in terms of the *relative JSJ decomposition* of the group pair (F, \underline{w}) . This is a canonical decomposition of the pair (F, \underline{w}) , which in a sense encodes all cyclic decompositions. The absolute version of this decomposition was described in the hyperbolic case by Bowditch [4]; the relative version in the free case was described by Cashen [13]. See also the work of Guirardel and Levitt, who explain how to construct this JSJ decomposition as a tree of cylinders [20].

Theorem 5.5 (Relative JSJ decomposition). *Let (F, \underline{w}) be an irreducible group pair. There is a canonical cyclic decomposition \mathcal{G} for F with the following properties.*

- (i) *The underlying graph of \mathcal{G} has three kinds of vertices – rigid, surface and cyclic – such that:*
 - (a) *if a vertex v is of rigid type then the induced pair at v is a rigid group pair;*
 - (b) *if a vertex v is of surface type then the induced pair at v is of surface type;*
 - (c) *if a vertex v is cyclic then the vertex group \mathcal{G}_v is (infinite) cyclic.*

- (ii) *The underlying graph of \mathcal{G} is bipartite, with red vertices cyclic and green vertices either rigid or surface. In particular, every edge adjoins exactly one cyclic vertex.*
- (iii) *Every peripheral subgroup $\langle w_i \rangle$ is conjugate into a unique cyclic vertex group. (These cyclic vertices are called peripheral.)*

The decomposition \mathcal{G} guaranteed by the theorem is called the *relative JSJ decomposition* of the pair (F, \underline{w}) . For an irreducible graph pair (Γ, \underline{w}) , we will refer to the disjoint union of the relative JSJ decompositions of the fundamental groups of the components as the *relative JSJ decomposition* of the pair (Γ, \underline{w}) .

We are now ready to state the main theorem of this section, which describes the relative JSJ decompositions of maximal ∂ -immersions.

Theorem 5.6. *If (Λ, \underline{u}) is locally irreducible and*

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \curlywedge (\Gamma, \underline{w})$$

is a maximal, admissible ∂ -immersion then the relative JSJ decomposition of (Λ, \underline{u}) has no rigid vertices. Furthermore, if there is such a maximal ∂ -immersion, then there is a maximal ∂ -immersion of surface type.

The proof is based on ideas from [42]; the main technical result of that paper is as follows [42, Theorem 8].

Theorem 5.7. *If (Γ, \underline{w}) is a rigid graph pair then there is a finite-sheeted cover $(\widehat{\Gamma}, \widehat{\underline{w}}) \rightarrow (\Gamma, \underline{w})$ such that, whenever a finite-sheeted cover $(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w})$ factors through $(\widehat{\Gamma}, \widehat{\underline{w}})$, the pair $(\Gamma, \underline{w}' \setminus w'_i)$ is irreducible for any component w'_i of \underline{w}' .*

We will also need a relative analogue of Shenitzer's lemma – see, for instance, [39, Corollary 1.1] – which we state here in the terminology of this paper.

Lemma 5.8 (Relative Shenitzer's lemma). *Consider a decomposition \mathcal{G} of a group pair (F, \underline{w}) . If the induced pair at every vertex of \mathcal{G} is irreducible then the pair (F, \underline{w}) is irreducible.*

We now assemble the lemmas that we will need to prove Theorem 5.6. The first shows how to use a rigid vertex to increase irreducible rank. Its proof is illustrated in Figure 3.

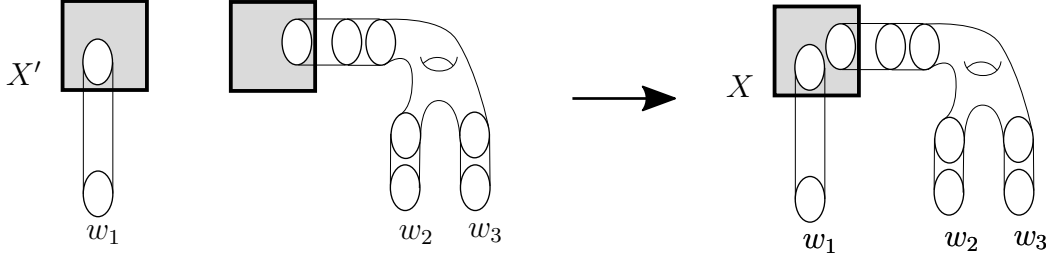


Figure 3: The graph of spaces X has one rigid vertex, one surface vertex, one non-peripheral cyclic vertex and three peripheral cyclic vertices. By taking multiple copies of X and deleting complementary components of the rigid vertices, we construct a new graph of spaces X' with greater projective \mathcal{P} -rank.

Lemma 5.9. *Let (Γ, \underline{w}) be an irreducible graph pair. If the relative JSJ of (Γ, \underline{w}) has a rigid vertex then there is a locally irreducible pair $(\Gamma', \underline{w}')$ and an admissible, essential map $(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w})$ with*

$$\frac{\chi_-(\Gamma')}{n(\underline{w}')} > \chi_-(\Gamma) .$$

Proof. Consider the relative JSJ decomposition \mathcal{G} of the pair (Γ, \underline{w}) . By Theorem 5.7 and Marshall Hall's theorem [22], after replacing (Γ, \underline{w}) with a finite-sheeted cover we may assume that every rigid vertex \mathcal{G}_u has the property guaranteed by Theorem 5.7. Note that, if $\nu(v)$ denotes the valence of the vertex v , then every rigid vertex u has $\nu(u) > 1$.

To construct Γ' , we realize the relative JSJ of (Γ, \underline{w}) as a graph of spaces X . We may take the vertex spaces of X to be graphs and the edge spaces to be circles, although this is not important for the subsequent argument. We now construct a new graph of spaces X' and an essential map $X' \rightarrow X$.

Let $k = \prod_u (\nu(u) - 1)$, where the product is taken over all rigid vertices of X . For each non-rigid vertex X_v , we take k copies of the induced pair at v . Consider a rigid vertex u , with incident edges $\{e_i\}$. For each edge $e_{i'}$ incident at u , we take $k/(\nu(u) - 1)$ copies of the pair $(X_u, \{X_{e_i} \mid i \neq i'\})$. Note that every edge space of X appears exactly k times in this collection of pairs. We may therefore glue the resulting collection of pairs up to form a graph of spaces X' .

By construction, X' is naturally equipped with a map $X' \rightarrow X$, which is

π_1 -injective by Proposition 1.7. Furthermore, X' is naturally equipped with exactly k copies of component of \underline{w} ; we call this map of circles \underline{w}_0 . Let Γ_0 be the disjoint unions of the cores (in the sense of [38]) of the covers of Γ corresponding to the components of X_0 . We may realize \underline{w}_0 as a collection of cycles in Γ_0 . Then $(\Gamma_0, \underline{w}_0) \rightarrow (\Gamma, \underline{w})$ is an admissible, essential map. The pair $(\Gamma_0, \underline{w}_0)$ is irreducible by Lemma 5.8, and hence by Lemma 2.11, can be unfolded to a locally irreducible pair $(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w})$.

Finally, we compute Euler characteristics. We have $n(\underline{w}') = k$, while

$$\frac{\chi_-(\Gamma')}{k} = \sum_u \frac{\nu(u)}{(\nu(u) - 1)} \chi_-(X_u) + \sum_v \chi_-(X_v) > \chi_-(\Gamma) ,$$

(where u ranges over all the rigid vertices of X and v ranges over all the non-rigid vertices). This completes the proof. \square

A similar argument shows that there are always maximal pairs of surface type.

Lemma 5.10. *Let (Γ, \underline{w}) be an irreducible graph pair. If the relative JSJ of (Γ, \underline{w}) has no rigid vertices then there is a locally irreducible group pair $(\Gamma', \underline{w}')$ of surface type and an admissible, essential $(\Gamma', \underline{w}') \rightarrow (\Gamma, \underline{w})$ with*

$$\frac{\chi_-(\Gamma')}{n(\underline{w}')} = \chi_-(\Gamma) .$$

Proof. Consider the JSJ decomposition \mathcal{G} of the pair (Γ, \underline{w}) . By Marshall Hall's theorem, we may assume that the attaching maps at cyclic vertices are all isomorphisms.

We realize \mathcal{G} as a graph of spaces X in the natural way, taking each surface vertex to be a compact surface and each cyclic vertex to be a circle, and define a new graph of spaces X' as follows. We take 2 copies of each surface vertex X_v . We take $\nu(w)$ copies of each non-peripheral cyclic vertex space X_w . We take 2 copies of each peripheral cyclic vertex space X_w of X ; these will each be a peripheral vertex of X' . Finally, we take $\nu(w) - 1$ further copies of each peripheral vertex space X_w ; these will be non-peripheral vertices of X' . It is now easy to see that we can assemble these to form X' so that every non-peripheral cyclic vertex group has exactly two incident edges and all the attaching maps are isomorphisms. As before, the natural map $X' \rightarrow X$ is π_1 -injective by Proposition 1.7.

Every non-peripheral cyclic vertex w is adjacent to exactly two surface vertices, and identified with two boundary components of these. We may therefore contract the two edges adjacent to w to obtain a larger surface vertex.

Thus, the resulting graph of spaces X' is homeomorphic to a surface Σ . The peripheral cyclic vertices equip X' with exactly two copies of each component of \underline{w} ; we call this collection of cyclic subgroups \underline{w}' . We note that $n(\underline{w}') = 2$ and that, since only surface vertices contribute to Euler characteristic, $\chi(\Sigma) = 2\chi(F)$. Therefore, if we replace (Σ, \underline{w}) by a locally irreducible graph pair $(\Gamma', \underline{w}')$ as in the previous lemma, $\chi_-(\Gamma')/n(\underline{w}') = \chi_-(\Gamma)$.

At this stage, we have a locally irreducible, admissible pair $(\Gamma', \underline{w}')$ of surface type, satisfying the required constraints on Euler characteristic, so that each component of \underline{w}' is conjugate into some component of $\partial\Sigma$ (and every component of $\partial\Sigma$ contains a component of \underline{w}'). To make this pair of surface type, we need \underline{w}' to be identified bijectively with $\partial\Sigma$. To ensure this, we first invoke Marshall Hall's theorem again, replacing Γ' with a finite-sheeted cover and \underline{w}' with its pullback, so that each component of \underline{w}' maps isomorphically to the component of $\partial\Sigma$ that contains it. For each component $\partial_i\Sigma \subseteq \partial\Sigma$, let n_i be the number of components of \underline{w}' contained in $\partial_i\Sigma$. Replacing Σ with two copies of itself, we may assume that each n_i is even. Without loss of generality, we may also assume that n_1 is minimal among the n_i . We now take n_1 copies of Σ , and equip each boundary component with exactly one component of \underline{w}' . We may then add annuli to Σ to pair up the remaining components of \underline{w}' . This completes the proof. \square

We can now apply these two lemmas to prove that we can always find a maximal ∂ -immersion of surface type.

Proof of Theorem 5.6. Suppose that the relative JSJ decomposition of (Λ, \underline{u}) has a rigid vertex. Lemma 5.9 applied to (Λ, \underline{u}) yields a locally irreducible pair $(\Lambda', \underline{u}')$ and an admissible, essential map $(\Lambda', \underline{u}') \rightarrow (\Lambda, \underline{u})$. The composition

$$(\Lambda', \underline{u}') \rightarrow (\Lambda, \underline{u}) \rightarrow (\Gamma, \underline{w})$$

is ∂ -essential and therefore, by Lemma 1.18, there is a ∂ -immersion

$$(\Lambda', \underline{u}') \rightarrow (\Lambda'_0, \underline{u}'_0) \looparrowright (\Gamma, \underline{w})$$

where $(\Lambda'_0, \underline{u}'_0) \looparrowright (\Gamma, \underline{w})$ is the canonical immersion associated to $(\Lambda', \underline{u}') \rightarrow (\Gamma, \underline{w})$. This ∂ -immersion then has greater projective \mathcal{P} -rank, which is a contradiction.

The second part of the theorem follows in the same way, using Lemma 5.10 instead of Lemma 5.9. \square

Our main technical theorem follows immediately.

Theorem 5.11. *Let (Γ, \underline{w}) be an irreducible graph pair. There exists a compact surface with boundary Σ and an admissible, essential map $(\Sigma, \partial\Sigma) \rightarrow (\Gamma, \underline{w})$.*

Proof. By Lemma 5.1, a maximal ∂ -immersion

$$(\Lambda, \underline{u}) \rightarrow (\Delta, \underline{v}) \looparrowright (\Gamma, \underline{w})$$

exists. By Theorem 5.6, there is such a maximal (Λ, \underline{u}) of surface type. The result now follows from Theorem 4.5. \square

6 Surface subgroups and hierarchies

In this section we deduce the claimed consequences of Theorem 5.11. We start with graphs of virtually free groups with virtually cyclic edge groups. The deduction of the existence of surface subgroups from a result like Theorem 5.11 is well known (cf. [7] or [42], for instance); we include an argument here for completeness.

Theorem 6.1. *Let Γ be the fundamental group of a graph of virtually free groups with virtually cyclic edge groups. If Γ is hyperbolic and one-ended then Γ contains a quasiconvex surface subgroup.*

Proof. By [43], Γ is residually finite and so virtually torsion-free. We may therefore assume that Γ is the fundamental group of a graph of free groups with cyclic edge groups. We call the vertices v of the underlying graph of Γ *non-cyclic*, and subdivide each edge, putting a *cyclic* vertex in the middle with vertex-group \mathbb{Z} .

Consider the induced pair (F_v, \underline{w}_v) for a non-cyclic vertex v . Since Γ is one-ended, (F_v, \underline{w}_v) is irreducible. By Theorem 5.11, we can replace each (F_v, \underline{w}_v) by an admissible, essential map of a surface pair $(\pi_1 \Sigma_v, \partial \Sigma_v)$. By

gluing these to the adjacent cyclic vertices, we define a new graph of free groups with cyclic edge groups \mathcal{H} , with every non-cyclic vertex of surface type. Note that $\chi(\Sigma_v) \leq 0$ for all v .

The fundamental group H of \mathcal{H} is equipped with a natural map $f : H \rightarrow \Gamma$, and by Proposition 1.7, f is injective. In particular, H contains no Baumslag–Solitar subgroups, since Γ is hyperbolic.

The graph of groups \mathcal{H} is a graph of surfaces glued along their boundary components to circles, which we realize in the natural way as a graph of spaces X . Note that $\chi(X) = \sum_v \chi(\Sigma_v) \leq 0$. By [43], after replacing H by a subgroup of finite index, we may assume that the attaching maps are all homeomorphisms. It is then easy to see that X can be thickened to a 3-manifold M with boundary. Since closed 3-manifolds have zero Euler characteristic,

$$\chi(\partial M) = 2\chi(M) = 2\chi(X) \leq 0$$

and so we may choose a component Σ of ∂M with $\chi(\Sigma) \leq 0$. Inclusion induces a natural map $\pi_1 \Sigma \rightarrow H$. But H is one-ended by Shenitzer’s lemma, and Dehn’s lemma then implies that the map $\pi_1 \Sigma \rightarrow H$ (and hence the composition $\pi_1 \Sigma \rightarrow \Gamma$) is injective.

Finally, Γ is locally quasiconvex [3, Theorem D], and hence the surface subgroup $\pi_1 \Sigma$ is quasiconvex. This completes the proof. \square

A group Γ is called *rigid* if it does not split over a (possibly finite) virtually cyclic subgroup. Given a group Γ , a *virtually cyclic hierarchy* for Γ is a set of subgroups of Γ obtained by passing to the vertex groups of a splitting of Γ over virtually cyclic edge groups, and then repeating this operation on those subgroups recursively. If a finite virtually cyclic hierarchy exists, terminating in (possibly finite) rigid subgroups, then we shall say that Γ *has a finite hierarchy*. Graphs of virtually free groups with virtually cyclic edge groups play a special role in the subgroup theory of groups that have finite hierarchies.

Remark 6.2. Let G be a one-ended group with a finite hierarchy, and let H be a one-ended subgroup in the hierarchy of G with no one-ended subgroups below it. Then either H is rigid or H is a graph of virtually free groups over virtually cyclic edge groups.

In [36], Sela showed that limit groups have a finite hierarchy. He also showed that a limit group without a \mathbb{Z}^2 subgroup is hyperbolic, and that non-abelian limit groups are never rigid. We thus obtain:

Corollary 6.3. *Every one-ended limit group contains a surface subgroup.*

Louder and Touikan showed that a hyperbolic group without 2-torsion has a finite hierarchy [29]. (The restriction on 2-torsion is technical, and conjecturally can be removed.) We thus obtain the following contribution towards the complete resolution of Gromov’s question.

Corollary 6.4. *Every one-ended hyperbolic group without 2-torsion either contains a surface subgroup or contains a quasiconvex, infinite, rigid subgroup.*

In particular, Gromov’s question is reduced to the rigid case (modulo the technical issue of 2-torsion).

7 Applications to profinite rigidity

In this section we discuss applications to Question 0.2 and related problems. As explained in [6, Theorem 4.17], Corollary 6.3 resolves the question for limit groups.

We include the proof for completeness. The key point is that it follows that the profinite completions of non-free limit groups have non-zero virtual second cohomology. We work with continuous cohomology, with coefficients in $\mathbb{Z}/2$.

Theorem 7.1. *If L is a limit group and not free then there is a subgroup L_0 of finite index in L such that $H^2(\widehat{L}_0; \mathbb{Z}/2) \neq 0$.*

Proof. By Corollary 6.3, L contains a subgroup S isomorphic to the fundamental group of a closed surface of non-positive Euler characteristic; in particular, $H^2(S) \neq 0$ (with coefficients in $\mathbb{Z}/2$). Since surface groups are *good* in the sense of Serre [19], it follows that the continuous cohomology $H^2(\widehat{S})$ is also non-zero. By [41], S is a virtual retract of L , so there is a finite-index subgroup L_0 containing S and a retraction $r : L_0 \rightarrow S$. Let $i : S \rightarrow L_0$ be the inclusion map, so $r \circ i = \text{id}_S$. Both r and i extend by continuity to maps $\hat{r} : \widehat{L}_0 \rightarrow \widehat{S}$ and $\hat{i} : \widehat{S} \rightarrow \widehat{L}_0$, and $\hat{r} \circ \hat{i} = \text{id}_{\widehat{S}}$. Therefore the induced maps on cohomology satisfy $\hat{i}^* \circ \hat{r}^* = \text{id}_{H^*(\widehat{S})}$. In particular, since $H^2(\widehat{S})$ is non-zero, $H^2(\widehat{L}_0)$ is also non-zero, as claimed. \square

Since every open subgroup of a profinite free group is profinite free, and hence has zero second cohomology, Corollary D follows immediately. Corollary E also follows quickly from Theorem 7.1

Proof of Corollary E. To prove the contrapositive, we assume that \underline{w} is not primitive in F . Recall that the *double* $D(\underline{w})$ is the fundamental group of the graph of groups with two vertices labelled by F , and one edge between them for each component of \underline{w} . The double $D(\underline{w})$ is a limit group and hence, by Theorem 7.1, has a subgroup D_0 of finite index with $H^2(\widehat{D}_0; \mathbb{Z}/2) \neq 0$. If \underline{w} were primitive in \widehat{F} then $\widehat{D(\underline{w})}$ would be free profinite, hence so would \widehat{D}_0 , and therefore $H^2(\widehat{D}_0; \mathbb{Z}/2)$ would be zero, a contradiction. \square

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